

DISTRIBUTION OF INCOME AND AGGREGATION OF DEMAND¹

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We show that, under certain regularity conditions, if the distribution of income is price independent and satisfies a condition on the shape of its graph, then total market demand, $F(p)$, is monotone; i.e., given two positive prices, p and q , one has $(p - q) \cdot (F(p) - F(q)) < 0$. These results allow for density functions increasing on some intervals, like unimodal distributions or even densities with more than one peak.

Similar assumptions on the distribution of endowments, yield a restricted monotonicity property on aggregate excess demand, where, now, wealth is determined by market prices. This property guarantees uniqueness and stability of equilibrium of the Walrasian pure exchange economy.

KEYWORDS: Law of demand, exchange economies, uniqueness of equilibrium, Weak Axiom of Revealed Preference, tatonnement, distribution of income.

1. INTRODUCTION

A SATISFACTORY THEORY LINKING THE CHARACTERISTICS of consumers identified by microeconomic theory and the properties of aggregated systems used in macroeconomic theory is still missing. The major stumbling block in filling the gap between these two fields is Sonnenschein's indeterminacy theorem (Sonnenschein (1971, 1973, 1974); it was later improved by McFadden, Mantel, Mas-Colell, Debreu, and others; see Shafer and Sonnenschein (1982)). This result states that any continuous mapping, defined on a compact set of prices, which satisfies Walras' law and is homogeneous of degree zero, coincides with the excess demand function of a certain economy.

This has some unpleasant consequences. One of these, is that it leads to the theoretical possibility of many equilibria. Furthermore, it questions the regularity (in terms of smoothness, like differentiability) of market demand and the stability of the equilibrium prices.

The problem of smoothness of market demand was studied by Dierker, Dierker, and Trockel (1984). The authors there show that conveniently dispersed distributions of preferences and wealth lead to a continuous or even C^1 demand function. The interested reader is referred to Trockel (1984) for a survey on market demand in large economies with nonconvex preferences and for instances in which aggregation has a smoothing effect.

Two main lines of research have dealt with the remaining issues, i.e., uniqueness and stability of equilibrium prices. One of them was initiated by

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Hildenbrand (1983) and developed by Härdle, Hildenbrand, and Jerison (1991). The gist of their method is to put restrictions on the *shape* of the distribution of *income*. The upshot of this approach is that one obtains that market demand is monotone. In particular, this implies that market demand for a particular commodity is a decreasing function of its price, and that the Weak Axiom of Revealed Preference holds for the aggregate.

However, the restrictions imposed are rather stringent. Firstly, income is price independent. Moreover, it is assumed that the distribution of income has a continuous nonincreasing density with bounded support.

With respect to the first assumption, it has been extended by Hildenbrand (1989a) and Hildenbrand and Kirman (1988) to the case in which individual endowments are collinear. The work of Chiappori (1985) has relaxed the second assumption to other types of densities for the case of identical consumers whose Engel curves have a specific functional form.

A second line of argument has been put forth by Grandmont (1987, 1992). His approach is to impose restrictions on the *shape of the agents' characteristics* rather than on the distribution of income. In this framework, the author makes use of a linear structure, the so called α -transform, on the space of demand functions.

As a result, the author proves very strong properties of the aggregate: market demand has a dominant diagonal Jacobian matrix and aggregate excess demand has the gross substitutability property. It follows then, that there is a unique equilibrium which is globally stable under the usual tatonnement process.

There are still some unsatisfactory features in the aforementioned viewpoint: One of them is that the way the α -transform is used does not have a straightforward economic interpretation.

The present work goes back to the first formulation. The key idea is again to control the “pathologies” of the income term in the Slutsky equation

$$DF(p) = S(p) - A(p),$$

where DF is the Jacobian matrix of mean demand, S is the average Slutsky compensated matrix, and A is the average income matrix. We do this by way of conditions on the shape of either income or initial endowments.

The novelty is twofold: First, in the case of price independent wealth, we allow for some increasing densities while the Law of Demand still holds for all prices. In fact, only a restriction on the *shape* of the density is required.

Secondly, for the case of pure market exchange economies, we obtain uniqueness of price equilibrium and local stability for this unique equilibrium, when initial endowments need not be necessarily collinear.

One further characteristic of this method, already present in Grandmont (1992), is that the matrix A does not need to be positive semidefinite, i.e., it may have some negative eigenvalues. Nevertheless, we do make use of the hypothesis that S is negative semidefinite.

The paper is organized as follows. First, we study the problem of demand aggregation in a model in which income is independent of prices. We introduce

the notion of metonymy, which appeared already in the context of Hardle, Hildenbrand, and Jerison (1991) and prove that if the shape of the density of the distribution of income satisfies a certain restriction, the Law of Demand holds for all prices.

We extend then these findings to the setting in which income is determined by the market price of initial endowments. The Law of Demand cannot hold in this case. However, a limited version of it can be proved. The Weak Axiom of Revealed Preference follows then for the aggregate economy and this yields the desired uniqueness and stability of the equilibrium price.

2. MARKET DEMAND

We consider an economy with n goods and a continuum of agents. Consumers will be distinguished from each other by their preferences and income, which for the moment will be assumed to be exogenously given.

DEFINITION 1: A *regular individual demand function* is a C^1 mapping $f: \mathbb{R}_{++}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$, satisfying the following properties:

- (i) Budget identity: $p \cdot f(p, w) = w$.
- (ii) Weak Axiom of Revealed Preference: for every (p, w) and (p', w') in $\mathbb{R}_{++}^n \times \mathbb{R}_+$ we have that $p' \cdot f(p, w) \leq w'$ implies $p \cdot f(p', w') \geq w$.
- (iii) The mapping $f(p, w)$ is continuously differentiable at each point $(p, w) \in \mathbb{R}_{++}^n \times \mathbb{R}_+$. Further, the Slutsky matrix $S(p, w)$ with entries given by

$$s_{ij} = \frac{\partial f_i}{\partial p_j} + \frac{\partial f_i}{\partial w} f_j$$

is symmetric.

Here $p \in \mathbb{R}_{++}^n$ denotes the vector of prices and $w \in \mathbb{R}_+$ is the consumer's budget. It follows from this definition that individual demand functions are homogeneous of degree 0 in (p, w) . Apart from differentiability of the demand functions, $f(p, w)$, the other conditions can be derived if one assumes they arise from continuous, strictly convex and nonsaturated preference relations.

The axiom of revealed preference as stated above implies that the Slutsky substitution matrix, $S(p, w)$, is negative semidefinite, i.e., for every $x \in \mathbb{R}^n$, $\langle S(p, w)x, x \rangle \leq 0$. The rank of the Slutsky matrix can be at most $n - 1$, since $S(p, w) \cdot p = 0$.

DEFINITION 2: A *regular market economy* is a triple, $\mathcal{E} = ((\mathcal{A}, \mu), f, \omega)$, where:

- (i) (\mathcal{A}, μ) is a Borel space with an atomless and regular measure, μ , assigning strictly positive measure to open subsets of \mathcal{A} .
- (ii) The mapping $\omega: \mathcal{A} \rightarrow \mathbb{R}_+$, which represents income level of consumers, is integrable and continuous.

(iii) The mapping $f: \mathcal{A} \times \mathbb{R}_{++}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ is continuous. It is also continuously differentiable in the last two arguments and, for each $\alpha \in \mathcal{A}$, $f(\alpha, \cdot, \cdot)$ is a regular individual demand function. Further, for each $p \in \mathbb{R}_{++}^n$, the functions $f(\cdot, p, \omega(\cdot))$, $(\partial f / \partial p_j)(\cdot, p, \omega(\cdot))$, and $(\partial f / \partial y)(\cdot, p, \omega(\cdot))$, defined on \mathcal{A} , are integrable.

In this context, at a given price system p , mean demand, $F(p)$, and mean income, $\bar{\omega}$, are defined by

$$F(p) = \int_{\mathcal{A}} f(\alpha, p, \omega(\alpha)) d\mu, \quad \bar{\omega} = \int_{\mathcal{A}} \omega(\alpha) d\mu.$$

The assumptions above imply that $F(p)$ is differentiable. Furthermore, for p in the interior of the positive orthant, we have

$$\frac{\partial F}{\partial p_j}(p) = \int_{\mathcal{A}} \frac{\partial f}{\partial p_j}(\alpha, p, \omega(\alpha)) d\mu < \infty.$$

The *Law of Demand* is said to hold for the economy E , if total demand $F(p)$ is monotone, i.e., if for each $p, q \in \mathbb{R}_{++}^n$, with $p \neq q$, one has that

$$(p - q) \cdot (F(p) - F(q)) < 0.$$

The Law of Demand is easily obtained if, for example, all individual demand functions are derived from homothetic preferences. Another instance in which the Law of Demand has been obtained, Hildenbrand (1983), is for economies with identical consumers and a decreasing density of income. These results have been extended by Chiappori (1985) to an economy with identical consumers in which the Engel curves can be written as $\sum_{k=0}^l g_k(p) \phi_k(w)$ and the functions $\phi_k(w)$ satisfy certain restrictions. Economies with agents not necessarily identical have been studied by Hildenbrand (1989a), again under the assumption of a decreasing density of income.

In this part we will be concerned with other situations in which the Law of Demand holds. Let $DF(p)$ denote the Jacobian matrix of F at p .

LEMMA 3: *Let G be a cone contained in \mathbb{R}_{++}^n ; suppose that for each $p \in G$ the Jacobian matrix $DF(p)$ is negative definite. Then, for all prices $p, q \in G$ we have that $(p - q) \cdot (F(p) - F(q)) < 0$.*

The proof is similar to the usual case (see Hildenbrand and Jerison (1989), Hildenbrand (1992)), so we will omit it here. We note, that the converse is not true.

For $w \in \mathbb{R}_+$, we let $B(w) = \{\alpha \in \mathcal{A} : \omega(\alpha) = w\}$ denote the set of consumers whose wealth is exactly w . The measure μ induces conditional distributions η_w on $B(w)$, for each w , and a probability measure, ν , on \mathbb{R}_+ such that for any

integrable function $h \in L^1(\mathcal{A}, d\mu)$ the following holds:

$$\int_{\mathcal{A}} h d\mu = \int_{\mathbb{R}_+} \left(\int_{B(w)} h d\eta_w \right) dv.$$

DEFINITION 4 (Härdle, Hildenbrand, and Jerison (1991)): The measure μ is said to be *metonymic* if:

(i) the probability measure ν has a density function $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is absolutely continuous on the interval $[0, I]$, where $I = \sup(\text{support}(\rho))$ (I may be infinity);

(ii) for each $z \in \mathbb{R}^n$ the following holds

$$\begin{aligned} \int_{\mathbb{R}_+} \left(\int_{B(w)} \frac{\partial}{\partial w} (f \cdot z)^2 d\eta_w \right) \rho(w) dw \\ = \int_{\mathbb{R}_+} \frac{\partial}{\partial w} \left(\int_{B(w)} (f \cdot z)^2 d\eta_w \right) \rho(w) dw. \end{aligned}$$

We refer the reader to Härdle, Hildenbrand, and Jerison (1991), Hildenbrand (1989a), and the next section for a discussion of this hypothesis.

For $\alpha \in \mathcal{A}$, $p \in \mathbb{R}_{++}^n$ the quadratic form

$$(2.1) \quad S(\alpha, p, x) = \sum_{i,j=1}^n s_{ij}(\alpha, p) p_i p_j x_i x_j,$$

is negative semidefinite and continuous. Here, $s_{ij}(\alpha, p)$ denotes the entries of the Slutsky substitution matrix of agent $\alpha \in \mathcal{A}$ corresponding to income $w(\alpha)$ and price system p .

Observe that for $x_0 = 1/\sqrt{n}(1, \dots, 1)$, we have $S(\alpha, p, x_0) = 0$ so the eigenvalues of $S(\alpha, p, x)$ have the form

$$(2.2) \quad \lambda_n(\alpha, p) \leq \lambda_{n-1}(\alpha, p) \leq \dots \leq \lambda_2(\alpha, p) \leq \lambda_1(\alpha, p) = 0.$$

Let

$$(2.3) \quad \lambda(\alpha) = \sup_{p \in \mathbb{R}_{++}^n} \lambda_2(\alpha, p) \leq 0, \quad \lambda = \int_{\mathcal{A}} \lambda(\alpha) d\mu,$$

and assume, momentarily, that $\lambda < 0$. Consider the quantity

$$\delta = \inf \left\{ \|\pi_0(x)\|^2 : \|x\| = 1, x \cdot x_0 \geq 0, \|x - x_0\| \geq \frac{\bar{\omega}}{n^{3/2}(2\bar{\omega} + 1)} \right\} > 0,$$

where π_0 denotes the orthogonal projection onto the plane $L = \{z: z_0 \cdot z = 0\}$. We can now define

$$(2.4) \quad M = \min \left\{ \frac{1}{2}, |\lambda| \delta \right\} > 0.$$

Hence, M depends on λ (which is obtained by aggregating the first nonzero eigenvalue of a certain matrix related to the Slutsky compensated matrix) and

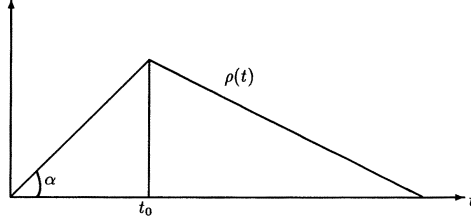


FIGURE 1

mean income. One may say, thus, that M reflects an aggregated feature of the “consumers’ tastes.” The next result shows that given this aggregated characteristic, one can find distributions for which the Law of Demand holds for all prices. These distributions have to satisfy only a certain restriction on the shape of the density $\rho(w)$.

THEOREM 5: *Let $\mathcal{E} = ((\mathcal{A}, \mu), f, \omega)$ be a regular market economy such that the measure μ is metonymic and $\lambda(\alpha) < 0$ on a set of positive measure. Let M be defined by (2.4). If*

$$(2.5) \quad \int_{\{w: \rho'(w) > 0\}} w^2 \rho'(w) dw < M,$$

then the Law of Demand holds for all prices $p, q \in \mathbb{R}_{++}^n$, with $p \neq q$.

We present next some examples to which this Theorem applies.

EXAMPLE 6: The density distribution is piecewise linear as given in Figure 1. It attains its peak at the point t_0 . Suppose we require that the total population be $\int_{\mathbb{R}_+} \rho(t) dt = 1$, whereas the fraction of it with income less than t_0 is $a < 1$. Assume also that, for that sector, the distribution ρ is given by $\rho(t) = \alpha t$. A simple computation shows that given M as in Theorem 5, we may take $t_0 = 3M/(2a)$, $\alpha = 8a^3/(9M^2)$. This distribution satisfies

$$\int_{\{\rho' > 0\}} t^2 \rho'(t) dt = M.$$

In particular, this example shows that the fraction of population for which $\rho(t)$ is increasing can be arbitrarily large. If this is the case, then $\rho(t)$ has to increase very slowly. Conversely, $\rho(t)$ can be very steep, as long as this occurs only for a small fraction of agents. Of course, the interplay between the numbers a and α is given by M .

EXAMPLE 7: It is possible to extend the above result to unimodal distributions. Namely, consider a unimodal distribution $\rho(w)$, which attains its peak at the point $t_0 \in \mathbb{R}$. Let $L = \sup \{\rho'(w): w \in [0, t_0]\}$. A simple calculation is required to show that condition 2.5 in Theorem 5 is verified if $Mt_0^3/3 < L$.

EXAMPLE 8: Let us consider an economy with the following structure:

(i) The space $(A, \mu) = (C, \eta) \times (D, \nu)$ is a product space, where (C, η) represents the “types” of the agents and $(D, \nu) = ([\omega_0, \omega_1], \rho)$ describes the distribution of income.

(ii) $\omega_0 = \inf\{\omega(\alpha): \alpha \in \mathcal{A}\} > 0$, $\omega_1 = \sup\{\omega(\alpha): \alpha \in \mathcal{A}\} < \infty$.

(iii) $\int_C \lambda(\alpha) d\eta \leq \lambda_0 < 0$.

Condition (i) above implies that μ will be metonymic as long as ρ is absolutely continuous. Condition (iii) says that the aggregate substitution effect is nonnegative in the economy for all positive prices.

Note that λ given by (2.3) equals now

$$\lambda = \int_{\omega_0}^{\omega_1} \left(\int_C \lambda(\alpha) d\eta \right) \rho(\omega) d\omega \leq \lambda_0 \int_{\omega_0}^{\omega_1} \rho(\omega) d\omega = \lambda_0$$

and does not depend on the distribution of income. Let M be defined as in (2.4), but where δ is given now by

$$\delta = \inf \left\{ \|\pi_0(x)\|^2 \|x\| = 1, \quad x \cdot x_0 \geq 0, \quad \|x - x_0\| \geq \frac{\omega_0}{n^{3/2}(2\omega_1 + 1)} \right\}$$

which clearly does not depend on ρ . Then Theorem 5 applies at once to all absolutely continuous distributions ρ which fulfill Equation 2.5.

As Examples 6 and 7 show it is now possible to find a whole family of metonymic measures μ , whose induced density of income distribution does not have to be *always decreasing*, and for which the Law of Demand holds. In addition, since (2.5) is given by an inequality, the set of density functions satisfying that restriction has a nonempty interior in the C^1 uniform topology.

There are other hypotheses which also guarantee that $\lambda(\alpha) < 0$ on a set of positive measure in Theorem 5. For example, as we will see in the next section, if the Slutsky substitution matrix has rank $n - 1$, where n is the number of commodities, then similar results follow.

3. MARKET EXCHANGE ECONOMIES

We apply next the above ideas to a competitive pure exchange economy to show that the same kind of distributional hypotheses (now restricting the *shape* of the distribution of endowments) imply uniqueness and stability of the equilibrium price.

In this new context, a *regular market exchange economy* will be a triple $((\mathcal{A}, \mu), f, \omega)$, where now $\omega \in L^1(\mathcal{A}, \mathbb{R}_+^n) \cap C(\mathcal{A}, \mathbb{R}_+^n)$ represents initial endowments, rather than income. In other words, we suppose that each consumer $\alpha \in \mathcal{A}$ has an endowment $\omega(\alpha) \in \mathbb{R}_+^n$. Otherwise, the functions f and ω satisfy the obvious modifications of conditions (i), (ii), and (iii) of Definition 2. Accordingly, in the present set up, each trader’s wealth is given by the market value of

his own endowment and is, therefore, price dependent. We let

$$F(p) = \int_{\mathcal{A}} f(\alpha, p, p \cdot \omega(\alpha)) d\mu, \quad \bar{\omega} = \int_{\mathcal{A}} \omega(\alpha) d\mu$$

denote, respectively, mean demand and endowment. Market excess demand is then

$$Z(p) = F(p) - \bar{\omega}.$$

It is immediately seen that $Z(p)$ is homogeneous of degree 0, bounded below by $\bar{\omega}$ and satisfies Walras' law: $p \cdot Z(p) = 0$. A positive price $p^* \in \mathbb{R}_{++}^n$ will be called an equilibrium price if $Z(p^*) = 0$.

We introduce next the concept of metonymy in this setting. For $s \in \mathbb{R}_+^n$ we let $G(s) = \{\alpha \in \mathcal{A} : \omega(\alpha) = s\}$ be the set of agents in the economy whose initial endowment is s . As in the preceding framework, the measure μ decomposes into conditional distributions, η_s on each $G(s)$ and ν on \mathbb{R}_+^n . Analysis of the Slutsky equation will be essential in the considerations below. Recall that it now reads

$$\frac{\partial f_i}{\partial p_j}(\alpha, p, p \cdot \omega(\alpha)) = S_{ij} \Big|_{(\alpha, p, p \cdot \omega(\alpha))} + (\omega_j(\alpha) - f_j) \frac{\partial f_i}{\partial w} \Big|_{(\alpha, p, p \cdot \omega(\alpha))}.$$

DEFINITION 9: We say that the measure μ is *metonymic* if the following conditions hold:

- (i) The probability measure ν has an absolutely continuous density function $g: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ with $g(s) = 0$ for all $s \in \partial \mathbb{R}_+^n$.
- (ii) For $k = 1, \dots, n$, for all $y \in \mathbb{R}^n$ and for all prices p ,

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \int_{G(s)} \frac{\partial}{\partial s_k} \langle s - f, y \rangle^2 d\eta_s g(s) ds \\ &= \int_{\mathbb{R}_+^n} \frac{\partial}{\partial s_k} \left(\int_{G(s)} \langle s - f, y \rangle^2 d\eta_s \right) g(s) ds. \end{aligned}$$

Condition (ii) in the above definition is fulfilled if the conditional distributions η_s do not depend on s . This is the case if $(\mathcal{A}, \mu) = (C \times D, \eta \otimes \nu)$ is a product space in which (C, η) describes the distribution of “types” of consumers and (D, ν) the allocation of initial endowments among each type. Thus, the simplest case in which metonymy holds is if the distribution of types is the same at all endowment levels, i.e., given a fixed endowment in the economy, say $s \in \mathbb{R}_+^n$, there are representatives of every “type” of consumer, drawn from C , with this endowment. In this sense, it points in the direction that all the possible types of consumers must be spread throughout all the levels of endowments present in the economy.

It is easy to verify that condition (ii) is equivalent to saying that for all prices p and all i, j, k ,

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \int_{G(s)} \frac{\partial}{\partial s_k} ((f_i - s_i)(f_j - s_j)) d\eta_s g(s) ds \\ &= \int_{\mathbb{R}_+^n} \frac{\partial}{\partial s_k} \left(\int_{G(s)} (f_i - s_i)(f_j - s_j) d\eta_s \right) g(s) ds. \end{aligned}$$

To clarify this condition, let us define, for $t, s \in \mathbb{R}_{++}^n$, the mappings

$$C_{ij}(t, s) = \int_{G(t)} (f_i(\alpha, p, p \cdot s) - s_i)(f_j(\alpha, p, p \cdot s) - s_j) d\eta_t.$$

That is, the functions $C_{ij}(t, s)$ would be the “average cross products of excess-demand functions” of the agents in the economy whose actual endowment is t , *had they been provided with an endowment s* . Metonymy says, precisely, that

$$\int_{\mathbb{R}_+^n} \frac{\partial}{\partial t_k} |C_{ij}(t, s)|_{t=s} g(s) ds = 0.$$

In particular it holds if, for example, $C_{ij}(\cdot, s)$ is constant near $t = s$. The mappings $(\partial C_{ij}(t, s) / \partial t_k)|_{t=s}$ represent the variation of these “averaged cross products of excess-demands of commodities” of agents, if one were allowed to “perturb” their actual endowment a little bit. Roughly speaking, metonymy can be interpreted, then, as saying that these local “perturbations” of the original endowments cancel out when aggregated by $g(s)$ and have no global effect on the economy.

It is a well known result (see Remark 13 below), that $F(p)$ cannot be monotone for all prices. However, in view of the results of the previous Section it is not unreasonable to expect the “Law of Demand” if one restricts prices to an appropriate set. More precisely, given a vector $e \in \mathbb{R}_+^n$, let $H(e) = \{x \in \mathbb{R}^n: x \cdot e = 0\}$ denote the plane perpendicular to e . The following is a modification of Lemma 6.1 in Hildenbrand and Kirman (1988).

LEMMA 10: *Let $F: \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ be a C^1 function, \mathcal{H} a cone contained in \mathbb{R}_{++}^n , and e any vector in \mathbb{R}_+^n . Suppose that for every $p \in \mathcal{H}$ the Jacobian matrix $DF(p)$ is negative definite on $H(e)$. Then $(p - q) \cdot (F(p) - F(q)) < 0$ for every $p, q \in \mathcal{H}$ with $p \neq q$ and $p \cdot e = q \cdot e$. The converse is not true.*

In order to exhibit the flexibility of the methods presented, we will consider now a slightly different variant of the hypotheses in Theorem 5. Namely, we suppose that, for all consumers, the Slutsky matrix has the maximum allowable rank, i.e., we assume it is exactly $n - 1$. In this case, the second eigenvalue $\lambda_2(\alpha, p)$ of $S(\alpha, p, x)$, given by 2.2, is strictly negative.

Consider a closed cone of prices $\mathcal{H} \subset \mathbb{R}_{++}^n$. Note that $\mathcal{H} \cap S_+^{n-1}$ is compact. Hence, letting

$$(3.1) \quad \lambda(\alpha) = \sup \{ \lambda_2(\alpha, p) : p \in \mathcal{H} \cap S_+^{n-1} \},$$

$$(3.2) \quad \lambda = \int_{\mathcal{H}} \lambda(\alpha) d\mu, \quad v = \inf_{p \in S_+^{n-1}} p \cdot \bar{\omega},$$

we see that $\lambda(\alpha)$ and λ are strictly negative. Thus, M defined by

$$(3.3) \quad M = \frac{|\lambda|v}{n\|\bar{\omega}\|_1} > 0,$$

is strictly positive. (We are using the notation $\|z\|_1 = \sum_{k=1}^n |z_k|$, for $z \in \mathbb{R}^n$.) Finally, for each $k = 1, \dots, n$, we let

$$B_k^+ = \left\{ s \in \mathbb{R}_{++}^n : \frac{\partial g(s)}{\partial s_k} > 0 \right\}.$$

We can now state a restricted version of the Law of Demand. This will be the key result to Theorem 14 below.

PROPOSITION 11: *Let $\mathcal{E} = ((\mathcal{A}, \mu), f, \omega)$ be a regular market economy such that the measure μ is metonymic and for each consumer, the Slutsky compensated matrix has rank $n - 1$. Fix a closed cone of prices $\mathcal{H} \subset \mathbb{R}_{++}^n$. Let $M > 0$ be defined by equation (3.3) and assume that for each $k = 1, \dots, n$,*

$$(3.4) \quad \int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \frac{M}{2}.$$

Then the excess demand function satisfies

$$(3.5) \quad (p - q) \cdot (Z(p) - Z(q)) < 0$$

for all prices $p, q \in \mathcal{H}$ with $p \neq q$ and $p \cdot \bar{\omega} = q \cdot \bar{\omega}$.

REMARK 12: The proof of Proposition 11 shows that the requirement that each consumer has preferences which yield a Slutsky matrix with rank $n - 1$ can be replaced by the following one: $\lambda(\alpha)$ as defined in equation (3.1) is strictly negative on a set of positive measure. And, in this case, the Theorem holds, as in the previous section, for all prices $p \neq q$ in \mathbb{R}_{++}^n with $p \cdot \bar{\omega} = q \cdot \bar{\omega}$.

REMARK 13: We observe as in Hildenbrand and Kirman (1988), that (3.5) cannot hold for all prices. Indeed, let $p \in \mathbb{R}_{++}^n$ such that $Z(p) \notin \mathbb{R}_{++}^n$. There is $q \in \mathbb{R}_{++}^n$ such that $q \cdot Z(p) < 0$. For $\lambda \in \mathbb{R}$ consider

$$\begin{aligned} (\lambda p - q)(Z(\lambda p) - Z(q)) &= \lambda p(Z(\lambda p) - Z(q)) \\ &\quad - q \cdot (Z(\lambda p) - Z(q)) \\ &= \lambda p(Z(p) - Z(q)) - q \cdot Z(p). \end{aligned}$$

For λ small enough the last term can be made positive.

We come now to the main result of this section. Our goal is to show that, for a wide family of exchange economies, there is unicity of equilibrium. As before, we will have two kinds of hypotheses: some regularity hypotheses and conditions similar to (3.4), restricting the shape of the function, which describe the distribution of initial endowments. In addition, we add hypothesis (ii) below to guarantee the existence of equilibrium prices. Let $\partial\mathbb{R}_{++}^n$ denote the boundary of \mathbb{R}_{++}^n , i.e., $\partial\mathbb{R}_{++}^n = \{p \geq 0: p_1 p_2 \cdots p_n = 0\}$.

THEOREM 14: *Let $\mathcal{E} = ((\mathcal{A}, \mu), f, \omega)$ be a regular market exchange economy satisfying:*

- (i) *The measure μ is metonymic.*
 - (ii) *If $\{p_n\}_{n=1}^\infty \subset \mathbb{R}_{++}^n$ converges to $p \in \partial\mathbb{R}_{++}^n$, then $\lim_{n \rightarrow \infty} \|F(p_n)\| = +\infty$.*
 - (iii) *For every consumer, the Slutsky compensated matrix has rank $n - 1$.*
- Then, there is $M > 0$ such that if, for all $k = 1, \dots, n$,*

$$\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \frac{M}{2},$$

there is a unique equilibrium price p^ . This equilibrium price is locally stable under the standard tatonnement process*

$$\dot{p}(t) = Z(p(t)).$$

Furthermore, there is closed cone of prices $\mathcal{H} \subset \mathbb{R}_{++}^n$, containing p^ , such that for all $q \in \mathcal{H}$, which is not collinear with p^* , we have that $p^* \cdot Z(q) > 0$ (the Weak Axiom of Revealed Preference for the aggregate holds in \mathcal{H}).*

EXAMPLE 15: We examine the higher dimensional analogue of Example 6. Let $g(s)$ be given by $g(s) = \prod_{i=1}^n g_i(s_i)$, where $g_i(s_i)$ has the form depicted in Figure 1 and $g_i(s_i) = \alpha_i t$, if g_i is increasing. It is clear that once M is given, one can adjust the α_i so that

$$\int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds \leq \frac{M}{2}.$$

Estimates based on higher dimensional unimodal distributions can also be used in Theorem 14. The computations will be omitted here.

EXAMPLE 16: Consider a regular market exchange economy $((\mathcal{A}, \mu), f, \omega)$ with the properties:

- (i) For every consumer $\alpha \in \mathcal{A}$, $\omega(\alpha) \gg 0$.
- (ii) \mathcal{A} is compact.
- (iii) Every consumer has nonsatiated preferences.
- (iv) If $\{p^k\}_{k=1}^\infty \subset \mathbb{R}_{++}^n$ is a sequence of positive prices such that, for some $i = 1, \dots, n$, $\lim_{k \rightarrow \infty} p_i^k = 0$, then the demand of every consumer $\alpha \in \mathcal{A}$ satisfies $\lim_{k \rightarrow \infty} f_i(\alpha, p^k, p^k \cdot \omega(\alpha)) = +\infty$.

Take the one-point compactification, $\mathbb{R}_*^n = \mathbb{R}_{++}^n \cup \{+\infty\}$ of \mathbb{R}_{++}^n . It follows from either condition (iii) or condition (iv) that for each $\alpha \in \mathcal{A}$, the map $f(\alpha, \cdot, \cdot): S_+^{n-1} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}^n$ has a continuous extension to ∂S_+^{n-1} , mapping ∂S_+^{n-1} to $+\infty$. Thus, we may consider each demand function $f(\alpha, \cdot, \cdot)$ as

belonging to the set C^0 consisting of continuous mappings $f: (S_+^{n-1} \cup \partial S_+^{n-1}) \times \mathbb{R}_{++} \rightarrow \mathbb{R}_*^n$. Endow the set C^0 with the topology of uniform convergence on compact sets (Dierker (1974, p. 120)). Let us assume also:

(v) The mapping $\mathcal{A} \rightarrow C^0$, which maps α to $f(\alpha, \cdot, \cdot)$ is continuous.

This corresponds to the idea that “similar agents behave similarly in similar situations” (Dierker (1974), Trockel (1984)). We will show next that for this space of consumers, it is possible to construct many density functions $g(s)$ which fulfill equation (3.4). For each price $p \in \mathbb{R}_{++}^n$ and commodity $i = 1, \dots, n$, define

$$g_i(p) = \inf \{f_i(\alpha, p, p \cdot \omega(\alpha)) : \alpha \in \mathcal{A}\}$$

which is homogeneous of degree 0 and strictly positive since preferences are nonsatiated and \mathcal{A} is compact. Note also that, from compactness, the infimum is actually a minimum, i.e., for each price $p \in \mathbb{R}_{++}^n$ there is $\alpha_p^i \in \mathcal{A}$ such that $g_i(p) = f_i(\alpha_p^i, p, p \cdot \omega(\alpha_p^i))$.

Suppose now that $\{p^k\}_{k=1}^\infty \subset \mathbb{R}_{++}^n$ converges to the boundary of \mathbb{R}_{++}^n , say $\lim_{k \rightarrow \infty} p_i^k = 0$. Since \mathcal{A} is compact, we may assume that $\{\alpha_{p^k}^i\}_{k=1}^\infty$ converges to, say, $\alpha^i \in \mathcal{A}$. Then,

$$\begin{aligned} \lim_{k \rightarrow \infty} g_i(p^k) &= \lim_{k \rightarrow \infty} f_i(\alpha_{p^k}^i, p^k, p^k \cdot \omega(\alpha_{p^k}^i)) \\ &= \lim_{k \rightarrow \infty} f_i(\alpha^i, p^k, p^k \cdot \omega(\alpha^i)) = +\infty. \end{aligned}$$

Thus, letting $g(p) = (g_1(p), \dots, g_n(p))$ and $\omega_{\max} = \sup \{\omega(\alpha) : \alpha \in \mathcal{A}\}$, there is a closed cone $\mathcal{H} \subset \mathbb{R}_{++}^n$ such that for each $p \in \mathbb{R}_{++}^n \setminus \mathcal{H}$ we have that $g(p) - \omega_{\max} \neq 0$.

Consider, now, a measure μ on \mathcal{A} . Then

$$F(p) = \int_{\mathcal{A}} f(\alpha, p, p \cdot \omega(\alpha)) d\mu \geq \int_{\mathcal{A}} g(p) d\mu = g(p),$$

$$\bar{\omega} = \int_{\mathcal{A}} \omega(\alpha) d\mu \leq \omega_{\max},$$

so, for any price $p \in \mathbb{R}_{++}^n \setminus \mathcal{H}$ and any measure μ on \mathcal{A} , we have that

$$F(p) - \bar{\omega} \geq g(p) - \omega_{\max} \neq 0,$$

i.e. all the possible equilibrium prices are contained in \mathcal{H} for any distribution of endowments μ . Let now

$$\omega_{\min} = \inf \{\omega(\alpha) : \alpha \in \mathcal{A}\} \gg 0, \quad v_0 = \inf_{p \in S_+^{n-1}} p \cdot \omega_{\min} > 0,$$

$$\lambda_0 = \sup \{\lambda(\alpha, p) : \alpha \in \mathcal{A}, p \in \mathcal{H} \cap S_+^{n-1}\} < 0.$$

None of these quantities depends on μ . But now Theorem 14 applies to all metonymic measures μ whose associated distribution $g(s)$ satisfies equation (3.4) with

$$M = \frac{\|\lambda_0\| v_0}{n \|\bar{\omega}_{\max}\|_1}$$

which clearly does not depend on $g(s)$. In particular, Example 15 shows that it is always possible to construct them.²

4. FINAL REMARKS

We have presented a model of a pure exchange economic system in which total market demand is monotone in prices. This allows one to obtain results on uniqueness and stability of the equilibrium price.

We have tried to obtain the above macroeconomic properties by means of not too unrealistic hypotheses on the distribution of total expenditure without demanding very restrictive conditions on each of the individual's preferences. In particular, our aim was to obtain as many testable hypotheses as possible. In this respect, we feel obliged to say that understanding the number M requires knowledge of the consumers' preferences and it may be very difficult to calculate in practice. Once this is done, it is possible to test whether a wide family of distributions of either income or endowments, including most of the working types, verify the requirements of Proposition 11. Thus, the main practical difficulty lies on the computation of the number M .

In the future, it is hoped that empirical estimates will be obtained in order to test whether the observable distributions satisfy the hypotheses proposed here.

To compare the present results with the line followed by Hildenbrand (1983), Hildenbrand (1989a), and Chiappori (1985), the novelty here is that we allow for some increasing densities of income without making assumptions on the Engel curves. Our methods, even though they are within the line started by the pioneering paper Hildenbrand (1983), can be further extended to the more general setting in which wealth is determined by the market price of initial endowments.

The alternative approach to the problem of aggregation of demand, followed by Grandmont (1992), makes somewhat strong assumptions on both the consumers preferences (aggregate desirability) and the distribution of "characteristics" (the conditional densities in each class of α -transform are all the same, and all the agents in that class have equal wealth). In return, the author obtains the remarkable result that one does not need to make any reference to individual rationality other than homogeneity and Walras' law.

As we have seen, in the present approach, the basic line of argument is to study the Slutsky equation and to show that one can control the pathologies introduced by the income effect matrix. Thus, it is not clear at all that our methods would still be applicable to an economy in which there is no individual rationality, as the ones studied by Grandmont (1987, 1992) and Quah (1993).

² It might be relevant to compare this result with the approach of Hildenbrand (1989b, 1992) and Grodal and Hildenbrand (1989). The author there demonstrates that if one assumes the existence of certain goods (so called "pure factors of production") from which the households derive income but not utility, along with some restrictions on individual demand functions, then the market demand function of such sectors does not "typically" satisfy the Weak Axiom of Revealed Preference. One of the assumptions is that none of the consumers ever demands positive amounts of the pure factors of production. Observe that this hypothesis is violated in the context of Theorem 14.

Which of these analyses will be, indeed, the one which finally allows us to derive the macroeconomic structure observed in markets from microeconomic principles remains to be unveiled by future research.

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APPENDIX

We will use the following conventional notation: $\mathbb{R}_+^l = \{z \in \mathbb{R}^l: z \geq 0, z \neq 0\}$, $\mathbb{R}_{++}^l = \{z \in \mathbb{R}^l: z_i > 0 \text{ for all } i = 1, \dots, l\}$. Similarly, we denote the usual inner product in \mathbb{R}^n by $\langle x, y \rangle = x \cdot y$. We will need the following norms in \mathbb{R}^n : $\|z\| = \sqrt{\sum_i z_i^2}$ and $\|z\|_1 = \sum_i |z_i|$. The inequality $\|z\|_1 \leq \sqrt{n} \|z\|$ is a standard result.

Let S_+^{n-1} denote the positive orthant of the unit sphere, $S_+^{n-1} = \{x \in \mathbb{R}_{++}^n: \|x\| = 1\}$, where n is the number of commodities.

For the proof of Theorem 5 we will make use of the following preliminary result.

LEMMA 17: *Let $g \in C^1(\mathbb{R}^n, \mathbb{R})$. Let G be a convex subset of \mathbb{R}^n and let $a, b \in G$ with $g(a) = 0$. Suppose for all $x \in G$, and $i = 1, \dots, n$, $\|\partial g(x)/\partial x_i\| < \alpha$. Then $\|g(b)\| < \alpha\sqrt{n}\|b - a\|$.*

PROOF OF LEMMA 17: Let $h(t) = g(bt + (1-t)a)$. Clearly, $h(0) = g(a) = 0$, $h(1) = g(b)$, so

$$\begin{aligned} \|g(b)\| &= \|h(1) - h(0)\| = \left\| \int_0^1 h'(t) dt \right\| \\ &= \left\| \sum_{i=1}^n (b_i - a_i) \int_0^1 \frac{\partial g}{\partial x_i} \Big|_{(bt + (1-t)a)} dt \right\| \\ &\leq \sum_{i=1}^n |b_i - a_i| \int_0^1 \left\| \frac{\partial g}{\partial x_i} \right\| dt \\ &< \alpha \sum_{i=1}^n \|b_i - a_i\| \leq \alpha\sqrt{n}\|b - a\|. \end{aligned} \quad Q.E.D.$$

PROOF OF THEOREM 5: From the remark following Lemma 3 we only have to show that the Jacobian matrix $DF(p)$ is negative definite. Let $x, y \in \mathbb{R}^n$ and define the quadratic forms

$$\begin{aligned} \bar{S}(p, y) &= \sum_{i,j=1}^n \int_{\mathcal{A}} s_{ij}(\alpha, p) y_i y_j d\mu, \\ \bar{A}(p, y) &= \sum_{i,j=1}^n \int_{\mathcal{A}} f_j \frac{\partial f_i}{\partial t} \Big|_{(\alpha, p, \omega(\alpha))} y_i y_j d\mu, \\ S(p, x) &= \sum_{i,j=1}^n \int_{\mathcal{A}} s_{ij}(\alpha, p) x_i x_j p_i p_j d\mu, \\ A(p, x) &= \sum_{i,j=1}^n \int_{\mathcal{A}} f_j \frac{\partial f_i}{\partial t} x_i x_j p_i p_j d\mu. \end{aligned}$$

With this notation and making use of the Slutsky equation, we see that $DF(p)$ is negative definite provided for each $y \in \mathbb{R}^n$, $y \neq 0$, we have that $\bar{S}(p, y) - A(p, y) < 0$. This will follow if we can show that for each $p \in \mathbb{R}_{++}^n$ and for each $x \in S^{n-1} = \{x \in \mathbb{R}^n: \|x\| = 1\}$

$$S(p, x) - A(p, x) < 0.$$

Because, given $y \in \mathbb{R}^n$, we let $z_i = y_i/p_i$, $x = (1/\|z\|)z \in S^{n-1}$. Then

$$\bar{S}(p, y) - \bar{A}(p, y) = S(p, z) - A(p, z) = \|z\|^2(S(p, x) - A(p, x)) < 0.$$

The eigenvalues of $S(\alpha, p, x)$ in (2.1) are given in equation (2.2). We can find an orthonormal transformation $T \in O(n)$, taking $x_0 = 1/\sqrt{n}(1, \dots, 1)$ to $(1, 0, \dots, 0)$ and such that if $U(\alpha, p)$ denotes the matrix associated to the quadratic form defined in (2.1) then $TU(\alpha, p)T^{-1}$ is the diagonal matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Let $y = Tx$ and denote by π and π_0 the orthogonal projections onto the planes $\{z: z_1 = 0\}$ and $\{z: x_0 \cdot z = 0\}$, respectively. Note that $T \circ \pi_0 = \pi \circ T$. For each $(\alpha, p) \in \mathcal{A} \times \mathbb{R}_{++}^n$, we have

$$\begin{aligned} S(\alpha, p, x) &= \langle U(\alpha, p)x, x \rangle = \langle UT^{-1}y, T^{-1}y \rangle = \langle TUT^{-1}y, y \rangle \\ &= \lambda_2(\alpha, p)y_2^2 + \cdots + \lambda_n(\alpha, p)y_n^2 \leq \lambda_2(\alpha, p)(y_2^2 + \cdots + y_n^2) \\ &\leq \lambda(\alpha)(y_2^2 + \cdots + y_n^2) = \lambda(\alpha)\langle \pi(y), \pi(y) \rangle \\ &= \lambda(\alpha)\langle \pi(Tx), \pi(Tx) \rangle = \lambda(\alpha)\langle T\pi_0(x), T\pi_0(x) \rangle \\ &= \lambda(\alpha)\langle \pi_0(x), \pi_0(x) \rangle = \lambda(\alpha)\|\pi_0(x)\|^2. \end{aligned}$$

Let λ be defined as in (2.3). Since $\lambda(\alpha) \leq 0$ and $\lambda(\alpha) < 0$ on a set of positive measure, we have that $\lambda < 0$. By integrating, we obtain

$$(5.1) \quad S(p, x) \leq \lambda\|\pi_0(x)\|^2.$$

Hence, letting $K = \bar{\omega}/(n^{3/2}(2\bar{\omega} + 1))$, we see that for all $p \in \mathbb{R}_{++}^n$ if $\|x - x_0\| \geq K$, $\|x\| = 1$, and $x \cdot x_0 \geq 0$, then

$$S(p, x) < -M.$$

Suppose now that ρ satisfies inequality (2.5). We focus first on the term $A(p, x)$. Observe that

$$\begin{aligned} (5.2) \quad A(p, x) &= \int_{\mathcal{A}} \left(\sum_j f_j x_j p_j \right) \left(\sum_i \frac{\partial f_i}{\partial t} x_i p_i \right) d\mu \\ &= \frac{1}{2} \int_{\mathcal{A}} \frac{\partial}{\partial t} \left(\sum_j f_j x_j p_j \right)^2 d\mu \\ &= \frac{1}{2} \int_{\mathbb{R}_+} \left(\int_{G(t)} \frac{\partial}{\partial t} \left(\sum_j f_j x_j p_j \right)^2 d\eta_t \right) \rho(t) dt. \end{aligned}$$

In particular, we obtain that $A(p, x_0) = \bar{\omega}/n$. Since the measure μ is metonymic, twice the last term in (5.2) equals

$$\int_{\mathbb{R}_+} \frac{\partial}{\partial t} \left(\int_{G(t)} \left(\sum_j f_j x_j p_j \right)^2 d\eta_t \right) \rho(t) dt$$

which, after integrating by parts, is the same as

$$N(p, x) - \int_{\mathbb{R}_+} \int_{G(t)} \left(\sum_j f_j x_j p_j \right)^2 d\eta_t \rho'(t) dt$$

where $N(p, x) = 0$ if the support of ρ is not bounded above or else

$$N(p, x) = \int_{G(b)} \left(\sum_j f_j x_j p_j \right)^2 \rho(b)$$

if $\sup(\text{support}(\rho)) = b \in \mathbb{R}$. In either case, $N(p, x) \geq 0$ and $\partial N(p, x)/\partial x_i \geq 0$ for $i = 1, \dots, n$. The boundary term at the lower end vanishes because $f(\alpha, p, 0) = 0$. Thus,

$$(5.3) \quad A(p, x) = \frac{1}{2} \left(N(p, x) - \int_{\rho' > 0} \int_{G(t)} \left(\sum_j f_j x_j p_j \right)^2 d\eta_t \rho' dt - \int_{\rho' \leq 0} \int_{G(t)} \left(\sum_j f_j x_j p_j \right)^2 d\eta_t \rho' dt \right).$$

We claim now that if $\|x - x_0\| < K$, then $A(p, x) > 0$. Indeed, since $\partial N(p, x)/\partial x_i \geq 0$, we must have from equation (5.3) that for $i = 1, \dots, n$,

$$\begin{aligned} \left| \frac{\partial A}{\partial x_i} \right| &\leq \int_{\rho' > 0} \int_{G(t)} \left| \sum_j f_j x_j p_j \right| f_i p_i d\eta_t \rho' dt - \int_{\rho' \leq 0} \int_{G(t)} \left| \sum_j f_j x_j p_j \right| f_i p_i d\eta_t \rho' dt + \frac{\partial N}{\partial x_i} \\ &\leq \int_{\rho' > 0} \int_{G(t)} \left(\sum_j f_j p_j \right) f_i p_i d\eta_t \rho' dt - \int_{\rho' \leq 0} \int_{G(t)} \left(\sum_j f_j p_j \right) f_i p_i d\eta_t \rho' dt + \frac{\partial N}{\partial x_i}. \end{aligned}$$

The budget identity implies that for each $\alpha \in \mathcal{A}$, $f(\alpha, p, t) \cdot p = t$. Since demand and prices are positive, for each j , we also have $f_j(\alpha, p, t) p_j \leq t$. Therefore

$$(5.4) \quad \begin{aligned} \left| \frac{\partial A}{\partial x_i} \right| &\leq \int_{\rho' > 0} \int_{G(t)} t^2 d\eta_t \rho' dt - \int_{\rho' \leq 0} \int_{G(t)} t^2 d\eta_t \rho' dt + b^2 \rho(b) \\ &= 2 \int_{\rho' > 0} \int_{G(t)} t^2 d\eta_t \rho' dt - \int_{\mathbb{R}_+} \int_{G(t)} t^2 d\eta_t \rho' dt + b^2 \rho(b) \\ &\leq 2M - \int_{\mathbb{R}_+} \int_{G(t)} t^2 d\eta_t \rho' dt + b^2 \rho(b) \\ &= 2\bar{\omega} + 2M \leq 2\bar{\omega} + 1, \end{aligned}$$

where we have used that

$$\begin{aligned} \bar{\omega} &= \int_{\mathcal{A}} \omega(\alpha) d\mu = \int_{\mathbb{R}_+} \int_{G(t)} t d\eta_t \rho(t) dt \\ &= \int_{\mathbb{R}_+} t \rho(t) dt = \frac{b^2 \rho(b)}{2} - \frac{1}{2} \int_{\mathbb{R}_+} t^2 \rho'(t) dt > 0. \end{aligned}$$

Equation (5.4) holds also for x in the open unit ball (we need this in order to make use of Lemma 17). Thus, if $A(p, x) = 0$, by applying Lemma 17, with $\alpha = 2\bar{\omega} + 1$ we obtain that

$$0 < A(p, x_0) = \frac{\bar{\omega}}{n} < (2\bar{\omega} + 1) \sqrt{n} \|x_0 - x\|.$$

So $\|x_0 - x\| > K$ and for x satisfying $\|x_0 - x\| < K$ we see that

$$S(p, x) - A(p, x) < 0.$$

Thus, we will assume from now on that $\|x - x_0\| \geq K$. The integral over the region $\{t: \rho'(t) \leq 0\}$ appearing in (5.3) is positive and will cause no problems. Accordingly, we will concentrate on the region $\{t: \rho'(t) > 0\}$. More precisely, we fix a price $p \in \mathbb{R}_{++}^n$ and decompose $A(p, x)$ into $A^+(p, x) + A^-(p, x)$, where

$$A^+(p, x) = -\frac{1}{2} \int_{\{t: \rho'(t) > 0\}} \int_{G(t)} \left(\sum_i f_i x_i p_i \right)^2 d\eta_t \rho'(t) dt,$$

$$A^-(p, x) = -\frac{1}{2} \int_{\{t: \rho'(t) \leq 0\}} \int_{G(t)} \left(\sum_i f_i x_i p_i \right)^2 d\eta_t \rho'(t) dt.$$

Notice that $A^-(p, x) \geq 0$. Observe also that the quadratic forms S and A satisfy $S(p, -x) = S(p, x)$ and $A(p, -x) = A(p, x)$. Thus, it is enough to show that $S(p, x) + A(p, x) < 0$ for x in the half sphere $S_0^{n-1} = \{z \in S^{n-1}: x_0 \cdot z \geq 0\}$. In this way, we may restrict ourselves to x lying on S_0^{n-1} and $\|x - x_0\| \geq K$. But then,

$$|A^+(p, x)| = \frac{1}{2} \int_{\rho' > 0} \int_{G(t)} \left(\sum_i f_i(\alpha, p, t) x_i p_i \right)^2 d\eta_t \rho' dt$$

$$\leq \int_{\rho' > 0} t^2 \rho' < M.$$

Hence, we see that $|S(p, x)| > A^+(p, x)$. Therefore,

$$S(p, x) - A(p, x) = S(p, x) - A^+(p, x) - A^-(p, x) - \frac{1}{2}N(p, x)$$

$$\leq S(p, x) - A^+(p, x) < 0$$

because $A^-(p, x) + \frac{1}{2}N(p, x) \geq 0$. And the proof is finished. Q.E.D.

We now start the preliminaries to prove Proposition 11. Let

$$(5.5) \quad H_p = \left\{ z \in \mathbb{R}^n: \sum_{i=1}^n p_i z_i \bar{\omega}_i = 0, \quad z_1^2 + \cdots + z_n^2 = 1 \right\}.$$

LEMMA 18: If $z \in H_p$ then $\|\pi_0(z)\| \geq 1/\sqrt{n}$.

PROOF OF LEMMA 18: First, $\pi_0(z) = z - (z \cdot x_0)x_0$, so $\|\pi_0(z)\|^2 = 1 - (z \cdot x_0)^2$. On the other hand, $z \cdot x_0 = \cos(z, x_0)$. Thus, the maximum of $(z \cdot x_0)^2$ is attained when the angle between z and x_0 is minimum, i.e. on the edges of the positive orthant. By symmetry we may assume that $z = (0, z_2, \dots, z_{n-1}, \sqrt{1 - \sum_{i=2}^{n-1} z_i^2})$. A simple computation shows that $z \cdot x_0$ attains its maximum, for $z_2 = \cdots = z_n = 1/\sqrt{n-1}$. So, $(z \cdot x_0)^2 \leq 1 - 1/n$ and the Lemma follows. Q.E.D.

PROOF OF PROPOSITION 11: By Lemma 10 it is enough to show that for each $p \in \mathcal{H}$, $DF(p)$ is negative semidefinite on $H(\bar{\omega})$. Since $DF(p)$ is homogeneous of degree -1 in p , it is enough to show this for $p \in \mathcal{H} \cap S_+^{n-1}$. Fix now a price $p \in \mathcal{H} \cap S_+^{n-1}$ and consider

$$S_{ij}(p) = \int_{\mathcal{A}} s_{ij}(\alpha, p) d\mu,$$

$$A_{ij}(p) = \int_{\mathcal{A}} (\omega_j(\alpha) - f_j) \frac{\partial f_i}{\partial t} \Big|_{(\alpha, p, p \cdot \omega(\alpha))} d\mu,$$

and

$$S(p, x) = \sum_{i,j=1}^n S_{ij}(p) p_i p_j x_i x_j, \quad \bar{S}(p, y) = \sum_{i,j=1}^n S_{ij}(p) y_i y_j,$$

$$A(p, x) = \sum_{i,j=1}^n A_{ij}(p) p_i p_j x_i x_j, \quad \bar{A}(p, y) = \sum_{i,j=1}^n A_{ij}(p) y_i y_j,$$

where, as before, $s_{ij}(\alpha, p)$ are the entries of the Slutsky's matrix corresponding to consumer $\alpha \in \mathcal{A}$ when his income is $p \cdot \omega(\alpha)$ and p are the prevailing prices.

Let $\lambda < 0$ be defined by (3.2). Then, by Lemma 18 and equation (5.1) in the proof of Theorem 5, it follows that, for $x \in H_p$,

$$S(p, x) \leq \frac{\lambda}{n}.$$

Note also, that

$$v = \inf_{p \in S_+^{n-1}} p \cdot \bar{\omega} = \inf_{p \in S_+^{n-1}} \|\bar{\omega}\| \cos(p, \bar{\omega}) > 0,$$

because it follows from metonymy that $\bar{\omega}$ is in the interior of \mathbb{R}_+^n .

We claim that if $x \in H_p$ then $S(p, x) + A(p, x) < 0$. Indeed, pick $x \in H_p$ and let $y_i = x_i p_i$, so $\bar{\omega} \cdot y = 0$. We define

$$a(p) = p \cdot \bar{\omega} \geq v > 0.$$

Firstly, we see that $a(p)S(p, x) < \lambda v/n$. On the other hand, we have that

$$\begin{aligned} a(p)A(p, x) &= a(p) \sum_{i,j} \int_{\mathcal{A}} (\omega_j - f_j) \frac{\partial f_i}{\partial t} y_i y_j d\mu \\ &= a(p) \int_{\mathcal{A}} \langle \omega - f, y \rangle \left\langle \frac{\partial f}{\partial t}, y \right\rangle d\mu \\ &= \sum_k \bar{\omega}_k \int_{\mathcal{A}} \langle \omega - f, y \rangle \left\langle p_k \frac{\partial f}{\partial t}, y \right\rangle d\mu \\ &= \sum_k \bar{\omega}_k \int_{\mathbb{R}_+^n} \int_{G(s)} \langle s - f, y \rangle \left\langle p_k \frac{\partial f}{\partial t}, y \right\rangle d\eta_s g(s) ds. \end{aligned}$$

Note that $\partial f(\alpha, p, p \cdot s) / \partial s_k = p_k (\partial f(\alpha, p, p \cdot s) / \partial t)$, so we can write the last term as

$$\begin{aligned} &\sum_k \bar{\omega}_k \int_{\mathbb{R}_+^n} \int_{G(s)} \langle s - f, y \rangle \left\langle \frac{\partial f}{\partial s_k}, y \right\rangle d\eta_s g(s) ds \\ &= \sum_k \bar{\omega}_k \int_{\mathbb{R}_+^n} \int_{G(s)} \left(y_k \langle s - f, y \rangle - \frac{1}{2} \frac{\partial}{\partial s_k} \langle s - f, y \rangle^2 \right) d\eta_s g(s) ds. \end{aligned}$$

But the first term vanishes because $\sum_k \bar{\omega}_k y_k = \bar{\omega} \cdot y = 0$.

Using now metonymy and integration by parts, we get

$$\begin{aligned} a(p)A(p, x) &= \frac{-1}{2} \sum_k \bar{\omega}_k \int_{\mathbb{R}_+^n} \int_{G(s)} \frac{\partial}{\partial s_k} \langle (s - f), y \rangle^2 d\eta_s g(s) ds \\ &= \frac{1}{2} \sum_k \bar{\omega}_k \int_{\mathbb{R}_+^n} \int_{G(s)} \langle (s - f), y \rangle^2 d\eta_s \frac{\partial g(s)}{\partial s_k} ds, \end{aligned}$$

and, as before, we have to worry only about the regions B_k^+ , $k = 1, \dots, n$. Accordingly, we define

$$A^+(p, x) = \frac{1}{2} \sum_k \bar{\omega}_k \int_{B_k^+} \int_{G(s)} \left(\sum_j (s_j - f_j) p_j x_j \right)^2 d\eta_s \frac{\partial g(s)}{\partial s_k} ds$$

and bound first the inner integrand. Note that

$$\begin{aligned}
& \left(\sum_j (s_j - f_j(\alpha, p, p \cdot s)) p_j x_j \right)^2 \\
&= \left(\sum_j s_j p_j x_j - \sum_j f_j(\alpha, p, p \cdot s) p_j x_j \right)^2 \\
&= \left(\sum_j s_j p_j x_j \right)^2 + \left(\sum_j f_j(\alpha, p, p \cdot s) p_j x_j \right)^2 - 2 \left(\sum_j s_j p_j x_j \right) \left(\sum_j f_j(\alpha, p, p \cdot s) p_j x_j \right) \\
&\leq \left(\sum_j s_j p_j \right)^2 + \left(\sum_j f_j(\alpha, p, p \cdot s) p_j \right)^2 + 2 \left(\sum_j s_j p_j \right) \left(\sum_j f_j(\alpha, p, p \cdot s) p_j \right) \\
&= 4(p \cdot s)^2
\end{aligned}$$

because $0 \leq x_i^2 \leq 1$ and $p \cdot f(\alpha, p, p \cdot s) = p \cdot s$. Since we are assuming $p \in S_+^{n-1}$, we obtain that $p \cdot s \leq \|s\|$. Consequently,

$$\begin{aligned}
A^+(p, x) &\leq 2 \sum_k \bar{\omega}_k \int_{B_k^+} \int_{G(s)} \|s\|^2 d\eta_s \frac{\partial g(s)}{\partial s_k} ds \\
&= 2 \sum_k \bar{\omega}_k \int_{B_k^+} \|s\|^2 \frac{\partial g(s)}{\partial s_k} ds.
\end{aligned}$$

Hence, if (3.4) holds, then

$$A^+(p, x) \leq M \sum_k \bar{\omega}_k = M \|\bar{\omega}\|_1 \leq |a(p)S(p, x)|.$$

It follows that

$$a(p)(S(p, x) + A(p, x)) \leq a(p)S(p, x) + A^+(p, x) < 0.$$

Therefore, $S(p, x) + A(p, x) < 0$, as advertised.

We apply now Lemma 10. Let $y \in \mathbb{R}^n$ such that $\sum_i y_i \bar{\omega}_i = 0$. Set $x_i = y_i / p_i$ and $z = (1/\|x\|)x \in H_p$. We conclude that

$$\bar{S}(p, y) + \bar{A}(p, y) = S(p, x) + A(p, x) = \|x\|^2 (S(p, z) + A(p, z)) < 0,$$

which finishes the proof of the theorem. Q.E.D.

PROOF OF THEOREM 14: It is a standard result (see Debreu (1982)), that if $Z(p)$ is continuous, bounded below, and satisfies Walras' law and property (iii), then there is $p^* \in \mathbb{R}_{++}^n$ such that $Z(p^*) = 0$.

By (ii), there is $\delta > 0$ such that if $d(p, \partial \mathbb{R}_{++}^n) \leq \delta$, then $Z(p) \neq 0$. Let $\mathcal{H} \subset \mathbb{R}_{++}^n$ be a closed cone, such that $d(p, \partial \mathbb{R}_{++}^n) \leq \delta$ whenever $p \in (\mathbb{R}_{++}^n \cap S_+^{n-1}) \setminus \mathcal{H}$. Then, the set of equilibrium prices is contained in \mathcal{H} .

It follows from Proposition 11, that there is M , such that if $g(s)$ fulfills equation (3.4) for $k = 1, \dots, n$, then the excess demand function satisfies $(p - q) \cdot (Z(p) - Z(q)) < 0$ for all prices $p \neq q \in \mathcal{H}$ with $(p - q) \cdot \bar{\omega} = 0$.

Fix an equilibrium price $p^* \in \mathcal{H}$, and let $q \in \mathcal{H}$, not collinear with p^* . Let $\lambda = q \cdot \bar{\omega} / p^* \cdot \bar{\omega} > 0$. Then $\lambda p^* \neq q$ and $(\lambda p^* - q) \cdot \bar{\omega} = 0$, so

$$-\lambda p^* \cdot Z(q) = (\lambda p^* - q) \cdot (Z(\lambda p^*) - Z(q)) < 0$$

since $Z(\lambda p^*) = Z(p^*) = 0$, $q \cdot Z(q) = 0$.

This implies that $p^* \cdot Z(q) > 0$ for any other $q \in \mathcal{H}$, not collinear with p^* . In particular, $Z(q) \neq 0$ for any other $q \in \mathcal{H}$, not collinear with p^* . Therefore, p^* is the only possible equilibrium price in \mathcal{H} and hence in \mathbb{R}_{++}^n . The proof that this equilibrium price is stable under the suggested tatonnement process is standard (Hildenbrand and Kirman (1988)) and will be omitted here. Q.E.D.

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